

All quantum expectation values as classical statistical mean values.

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Abstract

Given a physical quantum system described by a Hilbert space \mathcal{H} , for any bounded quantum observable (a bounded self-adjoint operator) T it is possible to define several "hidden observable" functions $f : \mathcal{H} \rightarrow \mathbb{R}$ associated to T and for any quantum mixed state (a density matrix) D it is possible to define several "hidden mixed states" (probability measures) μ on \mathcal{H} associated to D in such a way that the following equality is verified:

$$\text{Trace}[b(T) \cdot D] = \int_{\mathcal{H}} b(f(\psi)) \cdot d\mu(\psi)$$

whatever is the continuous function $b : \mathbb{R} \rightarrow \mathbb{R}$.

This formula gives a general way to express any expectation value computable in a quantum theory as a classical statistical mean value.

1. INTRODUCTION

This article is a mathematical paper giving another way to express all the expectation values and all the probabilities of a quantum theory but, in the same time, is a hidden variable theory avoiding all known no-go theorems.

For a better comprehension of our results we suggest a list of hypothesis to keep in mind; in the following we will constantly refer to a quantum system described by a separable, infinite-dimensional Hilbert space \mathcal{H} .

- (1) When we prepare the system in a given pure quantum state, given by a complex one-dimensional subspace, actually we prepare the system in a *hidden state* described by a non-zero vector ψ in the assigned complex line (the *apparent pure state*).

We suppose the existence of a measure $\eta_{\mathbb{C} \cdot \psi}$ on every complex line $\mathbb{C} \cdot \psi$ expressing the probability to find the apparent state $\mathbb{C} \cdot \psi$ in the hidden state ψ .

- (2) When we prepare a measure apparatus corresponding to a precise quantum observable, a self-adjoint operator T (here supposed bounded, for simplicity), actually we prepare a measure apparatus corresponding to a *hidden observable* described by a function $f : \mathcal{H} \rightarrow \mathbb{R}$ giving the values effectively observed.

The *apparent observable* T and the hidden observable f are joined by the condition:

$$\langle E_B^T \rangle_{\psi} = \eta_{\mathbb{C} \cdot \psi}(f^{-1}(B) \cap \mathbb{C} \cdot \psi)$$

expressing the equality between two ways to compute the probability that the observable will give a value falling in the borel subset B of \mathbb{R} for the apparent state $\mathbb{C} \cdot \psi$.

- (3) When we perform a quantum test, described by a projector E , actually we identify a subset L of \mathcal{H} (a *hidden test* or a *hidden proposition*); the hidden test L and the *apparent test* E are joined by the equality:

$$\langle E \rangle_\psi = \eta_{\mathbb{C} \cdot \psi}(L \cap \mathbb{C} \cdot \psi)$$

between two expressions of the probability that the test will receive an affirmative answer for the apparent state $\mathbb{C} \cdot \psi$.

- (4) When we prepare a mixed quantum state described by a density matrix D , actually we prepare the state in a hidden mixed state described by a probability measure μ on \mathcal{H} . The *apparent mixed state* D and the hidden mixed state μ are joined by the relation:

$$\text{Trace}[E \cdot D] = \mu(L)$$

for every couple of a hidden test L and the corresponding apparent test E , giving the equality between two expressions of the probability that the test will receive an affirmative answer for the apparent mixed state D .

- (5) When an apparent observable T acts on an apparent mixed state D actually a hidden observable f acts on a hidden mixed state μ and these objects are joined together by the condition:

$$\text{Trace}[T \cdot D] = \int f \cdot d\mu$$

expressing the equality between two ways to compute the expectation value for the observable acting on the apparent mixed state D

- (6) A hidden observable f corresponding to the self-adjoint operator T takes almost all its values in $\text{spec}[T]$, that is it can take values out of $\text{spec}[T]$ only on a subset of \mathcal{H} of measure zero; so changing the function f on a set of measure zero you can always get all its values in $\text{spec}[T]$.

It is important to declare that these results depend, in our opinion, vitally on the hypothesis that behind an apparent observable there are several hidden observables: we don't consider possible to choose a function for every self-adjoint operator in a reasonable way (cfr. section 3).

This places the present article strongly inside the contextual position: the experimental values observed depend not only on the variety of hidden states behind an apparent state but also on the existence of several hidden observables behind the apparent observable considered, each corresponding to a different experimental context (cfr. [K-S], Ghirardi in [B] 4.6.5, [G-D]).

In particular we don't consider the sum $f + g$ of two hidden observables f and g is, in general, again a hidden observable even when the observables correspond to compatible (commuting) operators. We push toward a reinforcement and a clarification of what should be called a "context": observable functions in the same context can be summed, functions in different contexts cannot and functions corresponding to non compatible observables are never in the same context.

This is compensated however by the possibility to find always summable hidden observables for compatible operators and more generally by the possibility to find an algebra of hidden functions corresponding to an assigned commutative algebra of operators.

In this way we avoid to fall in the hypothesis that bring to some no-go theorem (cfr. [F1], [F2], [K-S],[J-P],[M1],[M2],[P]).

The last section proves, under reasonable hypothesis, that the theory developped here is unique up to isomorphisms.

List of Symbols

ν_F = measure induced by the monotone function F : $\nu_F([a, b]) = F(b) - F(a)$
 $\varphi_*\mu$ = image measure of the measure μ via φ : $(\varphi_*\mu)(A) = \mu(\varphi^{-1}(A))$
 \tilde{F} = quasi-inverse function of the monotone function F : $\tilde{F}(s) = \text{Inf} \{r : F(r) \geq s\}$
 $b(T) = b \circ T$ = function of the bounded self-adjoint operator T
 $\text{spec}[T]$ = spectrum of the bounded self-adjoint operator T
 $\text{Trace}[T]$ = trace of the bounded self-adjoint operator T
 $\langle T \rangle_\psi$ = expectation value for the bounded self-adjoint operator T on ψ
 E_B^T = projector associated to the borel subset B in the spectral measure of T
 $\mathcal{B}_{sa}(\mathcal{H})$ = vector space of bounded self-adjoint operators of \mathcal{H}
 $DM(\mathcal{H})$ = the set of all density matrices on \mathcal{H}
 $PR(\mathcal{H})$ = the space of all orthogonal projection operators of \mathcal{H}

2. THE HIDDEN OBSERVABLES

From now on we will fix on the borelian subsets of the set \mathbb{C} of complex numbers a probability measure η without atoms and invariant by rotations.

For such a measure it is always possible to find borel maps $\varphi : \mathbb{C} \rightarrow]0, 1[$ such that $\varphi_*\eta = \lambda$ (where λ denotes the Lebesgue measure on $]0, 1[$) and subsets with any assigned measure between 0 and 1 (the space \mathbb{C} with the measure η is a standard nonatomic probability space).

Let $(\mathcal{H}, \langle, \rangle)$ be a separable Hilbert space over \mathbb{C} of infinite dimension, on every complex line $\mathbb{C} \cdot \psi$ (with $\|\psi\| = 1$) there is just one probability measure $\eta_{\mathbb{C} \cdot \psi}$ such that $\eta_{\mathbb{C} \cdot \psi}(B \cdot \psi) = \eta(B)$ for every borel subset B of \mathbb{C} . On \mathcal{H} we will consider the σ -algebra of subsets (called pseudo-borel subsets) A such that for every complex line $\mathbb{C} \cdot \psi$ the intersection $A \cap \mathbb{C} \cdot \psi$ is a borel subset of $\mathbb{C} \cdot \psi$. Correspondingly a map $f : \mathcal{H} \rightarrow \mathbb{R}$ will be called a pseudo-borel function if $f^{-1}(B)$ is a pseudo-borel subset of \mathcal{H} for every borel subset B of \mathbb{R} . A pseudo-borel subset A of \mathcal{H} will be called a zero measure subset if every intersection $A \cap \mathbb{C} \cdot \psi$ has measure zero in $\mathbb{C} \cdot \psi$.

We will use in the following $\mathcal{H} \setminus \{0\}$ as the total space of hidden states and each $\mathbb{C} \cdot \psi \setminus \{0\}$ as the set of hidden states behind each quantum state $[\psi]$ of the complex projective space $\mathbb{P}_{\mathbb{C}}(\mathcal{H})$; the constant addition we will make of the element 0 to these sets should not create confusion and it is made only with the hope to simplify the notations (if you prefer you can simply forget everywhere the element 0).

You can easily check that all the theory developed in this article works equally well if you consider as total hidden space a generic set Λ instead of $\mathcal{H} \setminus \{0\}$ and a partition of Λ in a family of subsets $\Lambda_{[\psi]}$ (where $[\psi]$ varies in $\mathbb{P}_{\mathbb{C}}(\mathcal{H})$) instead of the partition into the lines $\mathbb{C} \cdot \psi \setminus \{0\}$ of $\mathcal{H} \setminus \{0\}$, each $\Lambda_{[\psi]}$ furnished with a σ -algebra of subsets and a probability measure $\eta_{[\psi]}$ making $\Lambda_{[\psi]}$ a standard nonatomic probability space.

Definition 1. An essentially bounded pseudo-borel function $f : \mathcal{H} \rightarrow \mathbb{R}$ will be called a function with orthodox mean values if there exists a (unique) bounded self-adjoint operator T such that $\int_{\mathbb{C} \cdot \psi} f \cdot d\eta_{\mathbb{C} \cdot \psi} = \langle T \rangle_\psi$ for every $\psi \in \mathcal{H} \setminus \{0\}$.

The set \mathcal{F} of all functions with orthodox mean values is a real vector space; the self-adjoint operator associated to a function by the previous definition is uniquely determined. The map $\sigma : \mathcal{F} \rightarrow \mathcal{B}_{sa}(\mathcal{H})$ so defined is a real linear map.

We will use the symbol \mathcal{B} to denote the algebra of all real borel functions sending bounded subsets of \mathbb{R} in bounded subsets of \mathbb{R} . This algebra contains the constant functions, all the polynomials, all the continuous functions and is closed by compositions.

Note that for every bounded self-adjoint operator T all the operators $b \circ T$ (with b in \mathcal{B}) are well-defined bounded self-adjoint operators.

Definition 2. A function f with orthodox mean values will be called a (essentially bounded) hidden observable function on \mathcal{H} if for every function b in \mathcal{B} the composition $b \circ f$ has orthodox mean values and $\sigma(b \circ f) = b \circ \sigma(f)$.

We will denote by \mathcal{O} the set of all hidden observable functions on \mathcal{H} . Given a hidden observable f all functions $b \circ f$ (with b in \mathcal{B}) are hidden observable functions. A function g differing only on a zero-measure subset from an observable function f is also an observable function with $\sigma(f) = \sigma(g)$.

The sum or the product of hidden observable functions is not, in general, a hidden observable function.

If f is an observable function with $\sigma(f) = T$ then $E_B^T = \chi_B \circ T = \sigma(\chi_{f^{-1}(B)})$ for every borel subset B of \mathbb{R} and $\langle E_{(-\infty, s]}^T \rangle_\psi = \eta_{\mathbb{C} \cdot \psi}(f^{-1}(-\infty, s] \cap \mathbb{C} \cdot \psi)$ for every s in \mathbb{R} . That is the borel measure ν_{F_ψ} induced by the function $F_\psi(s) = \langle E_{(-\infty, s]}^T \rangle_\psi$ coincides with the image measure $(f|_{\mathbb{C} \cdot \psi})_* \eta_{\mathbb{C} \cdot \psi}$.

Theorem 1. An essentially bounded pseudo-borel function $f : \mathcal{H} \rightarrow \mathbb{R}$ is a hidden observable if and only if there exists a (unique) bounded self-adjoint operator T such that $\int_{\mathbb{C} \cdot \psi} f^n \cdot d\eta_{\mathbb{C} \cdot \psi} = \langle T^n \rangle_\psi$ for every $\psi \in \mathcal{H} \setminus \{0\}$ and every $n \geq 0$.

Proof. Obviously you have the equality: $\int_{\mathbb{C} \cdot \psi} b \circ f \cdot d\eta_{\mathbb{C} \cdot \psi} = \langle b \circ T \rangle_\psi$ for every polynomial function b ; to prove the same equality for a generic continuous function b consider a sequence $\{b_n\}_{n \geq 1}$ of polynomials uniformly converging to b on a closed interval $[-N, N]$ containing $\text{spec}[T]$; standard converging properties for the integrals and for the operators imply the desired equality.

Finally to prove the equality for a generic function b in \mathcal{B} consider a sequence $\{b_n\}_{n \geq 1}$ of continuous functions converging in the $L^1([-N, N])$ norm to b . \square

Definition 3. A pseudo-borel subset L of \mathcal{H} will be called a hidden proposition if its characteristic function is a hidden observable.

We will denote by \mathcal{L} the set of all hidden propositions of \mathcal{H} . The set \mathcal{L} is called the hidden logic of \mathcal{H} .

The empty set and \mathcal{H} are hidden propositions; the complement of a hidden proposition is again a hidden proposition. Every pseudo-borel zero-measure subset L of \mathcal{H} is a hidden proposition with $\sigma(\chi_L) = 0$.

The union or the intersection of two hidden propositions is not, in general, a hidden proposition.

Theorem 2. *Let L be a pseudo-borel subset of \mathcal{H} with χ_L in \mathcal{F} , the subset L is a hidden proposition if and only if the operator $\sigma(\chi_L)$ is a projector of \mathcal{H} .*

Proof. (\Rightarrow) If χ_L is an observable then $\sigma(\chi_L)^2 = \sigma(\chi_L^2) = \sigma(\chi_L)$

(\Leftarrow) Let $\sigma(\chi_L) = E$ be a projector. Whatever is b in \mathcal{B} we have: $b \circ \chi_L = [b(1) - b(0)] \cdot \chi_L + b(0) \cdot 1$, then taken $c(x) = [b(1) - b(0)] \cdot x + b(0)$ the function c is in \mathcal{B} and we have $\sigma(b \circ \chi_L) = [b(1) - b(0)] \cdot E + b(0) \cdot I = c \circ E = b \circ E$ since b and c take the same values on the spectrum of E (cfr. [W] ex. 7.36 pag. 210). \square

If L and M are hidden propositions then $L \subset M$ implies $\sigma(\chi_L) \leq \sigma(\chi_M)$; $\sigma(\chi_{L \cap M}) = I - \sigma(\chi_L)$; $L \cap M = \emptyset$ implies $\sigma(\chi_L) \cdot \sigma(\chi_M) = 0$. If $\{L_n\}_{n \geq 1}$ is a family of disjoint hidden propositions then $\bigcup_{n \geq 1} L_n$ is a hidden proposition with $\sigma(\chi_{\bigcup_{n \geq 1} L_n}) = \sum_{n \geq 1} \sigma(\chi_{L_n})$.

Theorem 3. *A function f with orthodox mean values is a hidden observable if and only if for every borel subset B of \mathbb{R} the subset $f^{-1}(B)$ is a hidden proposition.*

Proof. (\Rightarrow) $\chi_{f^{-1}(B)} = \chi_B \circ f$ is in \mathcal{F} for every borel subset B of \mathbb{R} and $\sigma(\chi_{f^{-1}(B)})^2 = \sigma(\chi_B^2 \circ f) = \sigma(\chi_B \circ f)$ is a projector.

(\Leftarrow) Since the family $\{f^{-1}(-\infty, s]\}_{s \in \mathbb{R}}$ is a family of hidden propositions essentially empty for s small and essentially \mathcal{H} for s big, the family $\{\sigma(\chi_{f^{-1}(-\infty, s]})\}_{s \in \mathbb{R}}$ is the spectral family of a bounded self-adjoint operator T .

Therefore: $\langle E_{(-\infty, s]}^T \rangle_\psi = \eta_{\mathbb{C} \cdot \psi}(f^{-1}(-\infty, s] \cap \mathbb{C} \cdot \psi)$ for every s in \mathbb{R} and every $\psi \neq 0$. In other words the borel measure ν_{F_ψ} induced by the function $F_\psi(s) = \langle E_{(-\infty, s]}^T \rangle_\psi$ coincides with the image measure $(f|_{\mathbb{C} \cdot \psi})_* \eta_{\mathbb{C} \cdot \psi}$. So we can compute: $\int_{\mathbb{C} \cdot \psi} b \circ f \cdot d\eta_{\mathbb{C} \cdot \psi} = \int_{\mathbb{R}} b \cdot d[(f|_{\mathbb{C} \cdot \psi})_* \eta_{\mathbb{C} \cdot \psi}] = \int_{\mathbb{R}} b \cdot d\nu_{F_\psi} = \langle b \circ T \rangle_\psi$ whatever is b in \mathcal{B} and we can state that all the functions $b \circ f$ have orthodox mean values with $\sigma(b \circ f) = b \circ T = b \circ \sigma(f)$. \square

Theorem 4. *For every self-adjoint bounded operator T it is possible to find a hidden observable f such that $\sigma(f) = T$.*

Proof. Let $\text{spec}[T] \subset [-A, A]$, for every $\psi \neq 0$ the monotone function $F_\psi(s) = \langle E_{(-\infty, s]}^T \rangle_\psi$ is 0 before $-A$ and 1 after $+A$, therefore its quasi-inverse \widetilde{F}_ψ is absolutely bounded by A and has the property: $(\widetilde{F}_\psi)_* \lambda_{]0, 1[} = \nu_{F_\psi}$ (cfr. [K and S] thm. 4 p. 94)

Let's fix for every complex line $\mathbb{C} \cdot \psi$ in \mathcal{H} a borel map $\gamma|_{\mathbb{C}^* \cdot \psi}: (\mathbb{C} \setminus \{0\}) \cdot \psi \rightarrow]0, 1[$ such that $(\gamma|_{\mathbb{C}^* \cdot \psi})_* \eta_{\mathbb{C} \cdot \psi} = \lambda_{]0, 1[}$. Therefore the function $f: \mathcal{H} \rightarrow \mathbb{R}$ defined by $f(\psi) = (\widetilde{F}_\psi \circ (\gamma|_{\mathbb{C}^* \cdot \psi}))(\psi)$ when ψ is in $(\mathbb{C} \setminus \{0\}) \cdot \psi$ and defined 0 in the vector 0 is absolutely bounded by A and it verifies: $(f|_{\mathbb{C} \cdot \psi})_* \eta_{\mathbb{C} \cdot \psi} = \nu_{F_\psi}$ for every line $\mathbb{C} \cdot \psi$. Proceeding as in the previous proof this implies that all the functions $b \circ f$ have orthodox mean values and moreover $\sigma(b \circ f) = b \circ T = b \circ \sigma(f)$. That is f is a hidden observable and $\sigma(f) = T$. \square

Remembering the definition of a quasi-inverse function, the observable f defined in the previous proof is given explicitly by the expression:

$$f_\gamma(\psi) = \min \left\{ r \in \mathbb{R} : \langle E_{(-\infty, r]}^T \rangle_\psi \geq \gamma(\psi) \right\}$$

We could prove that this expression is practically exhaustive: infact, assigned the operator T and a function f such that $\sigma(f) = T$, it is possible to find a map γ such that $f = f_\gamma$ (up to a zero measure set). However we will not present here the proof of this theorem since we will not need this property in the following.

Theorem 5. *Given a self-adjoint bounded operator T a hidden observable f such that $\sigma(f) = T$ modified on a set of measure zero verifies $\overline{f(\mathcal{H})} = \text{spec}[T]$.*

Proof. There exists a biggest open subset W of \mathbb{R} such that $f^{-1}(W)$ is a zero measure subset of \mathcal{H} . Since $f(\mathcal{H}) \setminus W$ is not empty we can redefine the function f on $f^{-1}(W)$ with a value chosen in $f(\mathcal{H}) \setminus W$; this new function f is again an observable with $\sigma(f) = T$ and moreover does not take values in W , that is does not allow non-empty open subsets U of \mathbb{R} with $f^{-1}(U)$ of zero measure. A value y of \mathbb{R} is not in $\text{spec}[T]$ if and only if $E_{]y-\varepsilon, y+\varepsilon[}^T = 0$ for a suitable $\varepsilon > 0$ that if and only if $f^{-1}]y - \varepsilon, y + \varepsilon[$ is a zero measure subset: but, for this new function f , this is equivalent to $f^{-1}]y - \varepsilon, y + \varepsilon[= \emptyset$ and to $y \notin \overline{f(\mathcal{H})}$. \square

Corollary 1. *For every projector E there exists a proposition L with $\sigma(\chi_L) = E$.*

Proof. Let f be an observable such that $\sigma(f) = E$ with $f(\mathcal{H}) \subset \text{spec}[E] = \{0, 1\}$, the function f is the characteristic function of the proposition $L = f^{-1}(\{1\})$. \square

Note that for every hidden observable function f the set $\mathcal{H} \setminus f^{-1}(\text{spec}[\sigma(f)])$ is a zero measure subset of \mathcal{H} .

3. ALGEBRAS AND CONTEXTS

Let's imagine to be able to build an apparatus suitable to measure one or several quantities of the hidden system in a deterministic way (that is you get for a given "observable" on a given "hidden state" always the same value); this defines a precise experimental context and a family \mathcal{C} of all possible "observable functions" on the total space of hidden states associated to that given experimental context.

In these hypothesis if you can measure $f(\psi)$ and $g(\psi)$ you can also compute $f(\psi) + g(\psi)$, $f(\psi) \cdot g(\psi)$ and $k \cdot f(\psi)$ for every constant k , so \mathcal{C} must be a commutative algebra of functions. Moreover nothing can prevent you to compute $b(f(\psi))$ where b is any available real function, therefore it is not restrictive to suppose \mathcal{C} also closed with respect to the composition with the functions b of \mathcal{B} . So this kind of algebra is, at least from a mathematical viewpoint, representative of the choice of an experimental context.

Example 1. Let $\{L_n\}_{n \geq 1}$ be a family of (non-zero measure) pairwise disjoint hidden propositions of \mathcal{H} . Let's define:

$$\mathcal{C} = \left\{ f : \mathcal{H} \rightarrow \mathbb{R}; \quad f = \sum_{n \geq 1} c_n \cdot \chi_{L_n} \text{ with } \{c_n\}_{n \geq 1} \text{ bounded} \right\}$$

the family \mathcal{C} is an algebra of hidden observable functions closed by the compositions with the borel functions b in \mathcal{B} . The image of \mathcal{C} via σ is the commutative algebra

of bounded self-adjoint operators:

$$\mathcal{A} = \left\{ T; \quad T = \sum_{n \geq 1} c_n \cdot \sigma(\chi_{L_n}) \text{ with } \{c_n\}_{n \geq 1} \text{ bounded} \right\}$$

and $\sigma| : \mathcal{C} \rightarrow \mathcal{A}$ is an isomorphism of algebras.

The following theorem puts a strong limit to the existence of such mathematical objects \mathcal{C} .

Theorem 6. *Let \mathcal{C} be an algebra of functions on \mathcal{H} closed with respect to the composition with the functions of \mathcal{B} , the following alternative holds:*

- \mathcal{C} is not contained in \mathcal{O}
- or
- $\sigma(\mathcal{C})$ is a commutative family of bounded self-adjoint operators.

Proof. Let's suppose $\mathcal{C} \subset \mathcal{O}$; let's take two functions f and g in \mathcal{C} and two borel subsets A and B of \mathbb{R} . Since f and g are hidden observables the subsets $L = f^{-1}(A)$ and $M = g^{-1}(B)$ are two hidden propositions in \mathcal{H} with $\sigma(\chi_L) = \sigma(\chi_A \circ f) = \chi_A \circ \sigma(f) = E_A^{\sigma(f)}$ and $\sigma(\chi_M) = E_B^{\sigma(g)}$. Moreover, since \mathcal{C} is closed with respect to the composition with the functions of \mathcal{B} , the functions $\chi_L = \chi_A \circ f$ and $\chi_M = \chi_B \circ g$ are in \mathcal{C} .

Let $h : \mathcal{H} \rightarrow \{1, 2, 3, 4\}$ be the function taking value 1 on the elements of $L \setminus M$, value 2 on the elements of $M \setminus L$, value 3 on the elements of $L \cap M$ and value 4 on the set $\mathcal{C}L \cap \mathcal{C}M$. Since $\chi_{h^{-1}(1)} = \chi_L - \chi_L \cdot \chi_M$, $\chi_{h^{-1}(2)} = \chi_M - \chi_L \cdot \chi_M$, $\chi_{h^{-1}(3)} = \chi_L \cdot \chi_M$, $\chi_{h^{-1}(4)} = 1 - \chi_L - \chi_M + \chi_L \cdot \chi_M$ and $\chi_{h^{-1}\{n_1, \dots, n_r\}} = \sum \chi_{h^{-1}(n_i)}$ (when n_1, \dots, n_r are distinct numbers in $\{1, 2, 3, 4\}$), all the possible characteristic functions $\chi_{h^{-1}\{n_1, \dots, n_r\}}$ are in the algebra \mathcal{C} and by hypothesis in \mathcal{O} . Therefore h is a hidden observable with $L = h^{-1}(\{1, 3\})$ and $M = h^{-1}(\{2, 3\})$; then $E_A^{\sigma(f)} = \sigma(\chi_L) = E_{\{1,3\}}^{\sigma(h)}$ and $E_B^{\sigma(g)} = \sigma(\chi_M) = E_{\{2,3\}}^{\sigma(h)}$ must commute as projectors in the same spectral measure. For the arbitrariness of A and B the operators $\sigma(f)$ and $\sigma(g)$ commute. \square

The theorem just proved is a kind of no-go theorem since it claims that you cannot hope to find a hidden variable theory where you can realize an algebra of hidden observable functions associated with an "experimental context" and representing non-commuting bounded self-adjoint operators.

Alternatively you could say that such an "experimental context" can be imagined but with some of its associated functions out of \mathcal{O} , that is functions whose mean values have a non-ortodox behaviour (precisely functions f such that whatever is T bounded self-adjoint operator there exists a non-zero vector ψ and a non-negative integer n with $\int_{\mathcal{C} \cdot \psi} f^n \cdot d\eta_{\mathcal{C} \cdot \psi} \neq \langle T^n \rangle_\psi$).

If you don't ask to represent non-commuting bounded self-adjoint operators you have a positive answer:

Theorem 7. *Let \mathcal{A} be a commutative algebra of bounded self-adjoint operators closed by the compositions with the functions of \mathcal{B} , it is possible to find an algebra \mathcal{C} of hidden observable functions closed by the compositions with the functions of \mathcal{B} such that:*

- (1) $\sigma(\mathcal{C}) = \mathcal{A}$
- (2) $\sigma|_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{A}$ is an algebra homomorphism
- (3) $\sigma(h) = 0$ for h in \mathcal{C} if and only if h is zero out of a pseudo-borel null set.

Proof. Let $\mathcal{A} = \{A_i\}_{i \in I}$, since \mathcal{H} is separable and all the operators A_i commute each other there exists a self-adjoint operator T_0 (may be not bounded) and a family of borel functions $\{b_i\}_{i \in I}$ such that $A_i = b_i \circ T_0$ for every i in I (cfr. [VN] and [V]). Since all the operators A_i are bounded it is possible to correct the borel functions $\{b_i\}_{i \in I}$ and take them all bounded. Let $\mathcal{A}_1 = \mathcal{B} \circ T_0$ this is a commutative algebra of self-adjoint operators containing \mathcal{A} .

Proceedings as in the proof of theorem 4 it is possible to find a pseudo borel function $f_0 : \mathcal{H} \rightarrow \mathbb{R}$ with $(f_0|_{\mathbb{C} \cdot \psi})_* \eta_{\mathbb{C} \cdot \psi} = \nu_{F_\psi^T}$ for every complex line $\mathbb{C} \cdot \psi$ (where $F_\psi^{T_0}(s) = \langle E_{(-\infty, s]}^{T_0} \rangle_\psi$ and $\nu_{F_\psi^{T_0}}(B) = \langle E_B^{T_0} \rangle_\psi$ for every borel subset B in \mathbb{R}).

We can prove that the function $b \circ f_0$ is a hidden observable for every b in \mathcal{B} ; in fact $b_* f_{0*} \eta_{\mathbb{C} \cdot \psi}(B) = \langle E_{b^{-1}B}^{T_0} \rangle_\psi = \langle E_B^{b \circ T_0} \rangle_\psi$ for every borel subset B in \mathbb{R} , that is $b_* f_{0*} \eta_{\mathbb{C} \cdot \psi} = \nu_{F_\psi^{b \circ T_0}}$ therefore: $\int_{\mathbb{C} \cdot \psi} b \circ f_0 \cdot d\eta_{\mathbb{C} \cdot \psi} = \int_{\mathbb{R}} id \cdot d(b_* f_{0*} \eta_{\mathbb{C} \cdot \psi}) = \int_{\mathbb{R}} id \cdot d\nu_{F_\psi^{b \circ T_0}} = \langle b \circ T_0 \rangle_\psi$. This proves that every $b \circ f_0$ has orthodox mean values and also that $\sigma(b \circ f_0) = b \circ T_0$ for every b in \mathcal{B} ; this proves that every $b \circ f_0$ is a hidden observable.

Let $\mathcal{C}_1 = \mathcal{B} \circ f_0$ this is an algebra of hidden observable funtions closed by the compositions with the functions of \mathcal{B} , the points 1. and 2. follow immediately for \mathcal{A}_1 and \mathcal{C}_1 . If $\sigma(b \circ f_0) = 0$ then $\eta_{\mathbb{C} \cdot \psi}((b \circ f_0)^{-1}(\mathbb{R} \setminus \{0\}) \cap \mathbb{C} \cdot \psi) = 0$ for every complex line $\mathbb{C} \cdot \psi$ and $(b \circ f_0)^{-1}(\mathbb{R} \setminus \{0\})$ is a pseudo-borel null subset of \mathcal{H} . This proves also the point 3.

To conclude the proof let's consider the algebra $\mathcal{C} = (\sigma|_{\mathcal{C}_1})^{-1}(\mathcal{A})$, if $f = b \circ f_0$ is in \mathcal{C} and c is in \mathcal{B} then $c \circ f$ is in \mathcal{C}_1 with $\sigma(c \circ f) = c \circ \sigma(f)$ in \mathcal{A} and therefore $c \circ f$ is in \mathcal{C} . \square

In the example given above the inverse map $\Phi = (\sigma|_{\mathcal{C}})^{-1} : \mathcal{A} \rightarrow \mathcal{C} \subset \mathcal{O}$ is a map with the property $\Phi(b \circ T) = b \circ \Phi(T)$ for every b in \mathcal{B} and every T in \mathcal{A} and express the possibility, at least from a mathematical viewpoint, to fix an "experimental context" where to measure deterministic observables corresponding one-to-one to the observables in \mathcal{A} .

A map of this kind extended as much as possible would give a general family of "experimental contexts" pacifically coexisting. But it is not possible to go too far in this direction:

Theorem 8. *Let \mathcal{D} be a subset of $\mathcal{B}_{sa}(\mathcal{H})$ closed by the compositions with the functions of \mathcal{B} and $\Phi : \mathcal{D} \rightarrow \mathcal{O}$ a map such that: $\Phi(b \circ T) = b \circ \Phi(T)$ for every b in \mathcal{B} and every T in \mathcal{D} , then \mathcal{D} is properly contained in $\mathcal{B}_{sa}(\mathcal{H})$.*

Proof. Let's suppose, conversely, that such a map $\Phi : \mathcal{B}_{sa}(\mathcal{H}) \rightarrow \mathcal{O}$ exists. For every projector E let $f_E = \Phi(E)$, since $f_E^2 = \Phi(E)^2 = \Phi(E^2) = \Phi(E) = f_E$ the

function f_E is a characteristic function with $f_E = \chi_{L_E}$ where L_E is the hidden proposition $L_E = f_E^{-1}(1)$.

In particular $\Phi(I) = \chi_L$; if $L \subsetneq \mathcal{H}$ let's take $\varphi \notin L$ and a function b in \mathcal{B} with $b(0) \neq 0$. We have $\Phi(b \circ I)(\varphi) = \Phi(b(1) \cdot I)(\varphi) = \Phi[(b(1) \cdot id_{\mathbb{R}}) \circ I](\varphi) = [(b(1) \cdot id_{\mathbb{R}}) \circ \chi_L](\varphi) = 0$ but $(b \circ \chi_L)(\varphi) = b(0) \neq 0$.

Therefore $L = \mathcal{H}$ and $\Phi(I) = 1$.

Let's fix once for all a unit vector ψ_0 and let's define the map $G : \mathbb{S}(1) \rightarrow \{0, 1\}$ given by $G(\psi) = \chi_{L_{E[\psi]}}(\psi_0)$ where $E[\psi]$ is the orthogonal projector on the line $\mathbb{C} \cdot \psi$.

Let's consider an orthonormal base $\{\psi_n\}_{n \geq 1}$ of \mathcal{H} ; if $\{a_n\}_{n \geq 1}$ is a bounded injective sequence of real numbers the operator $T = \sum_{n \geq 1} a_n \cdot E[\psi_n]$ is a bounded self-adjoint operator with $E[\psi_n] = E_{\{a_n\}}^T = \chi_{\{a_n\}} \circ T$ for every $n \geq 1$. Therefore taken $f = \Phi(T)$ we get: $\chi_{f^{-1}(\{a_n\})} = \chi_{\{a_n\}} \circ f = \chi_{\{a_n\}} \circ \Phi(T) = \Phi(\chi_{\{a_n\}} \circ T) = \Phi(E_{\{a_n\}}^T) = \Phi(E[\psi_n]) = \chi_{L_{E[\psi_n]}}$ that is $L_{E[\psi_n]} = f^{-1}(\{a_n\})$.

Since $I = \sum_{n \geq 1} E[\psi_n] = \sum_{n \geq 1} E_{\{a_n\}}^T = E_{\cup\{a_n\}}^T = \chi_{\cup\{a_n\}} \circ T$ we have $1 = \Phi(I) = \chi_{\cup\{a_n\}} \circ f = \sum \chi_{f^{-1}(\{a_n\})}$ that is $\{L_{E[\psi_n]}\}$ is a partition of \mathcal{H} and the vector ψ_0 lies exactly in one of the sets $\{L_{E[\psi_n]}\}$.

Therefore $\sum_{n \geq 1} G(\psi_n) = 1$ for every orthonormal base $\{\psi_n\}_{n \geq 1}$, then G is, by definition (cfr. [G]), a Gleason frame function of weight 1. Since $\dim(\mathcal{H}) \geq 4$ there exists a bounded self-adjoint operator S such that $\langle S \rangle_{\psi} = G(\psi)$ for every ψ in $\mathbb{S}(1)$; the continuity of $\langle S \rangle$ implies $S = 0$ or $S = I$ and then $G = 0$ or $G = 1$. In both cases we don't have $\sum_{n \geq 1} G(\psi_n) = 1$ for an orthonormal base $\{\psi_n\}_{n \geq 1}$: contradiction. \square

4. THE HIDDEN MIXED STATES

Definition 4. A probability measure μ defined on the pseudo-borel subsets of \mathcal{H} will be called a hidden mixed state on \mathcal{H} if for every couple of hidden propositions L and M we have $\mu(L) = \mu(M)$ when $\eta_{\mathbb{C} \cdot \psi}(L \cap \mathbb{C} \cdot \psi) = \eta_{\mathbb{C} \cdot \psi}(M \cap \mathbb{C} \cdot \psi)$ for every complex line $\mathbb{C} \cdot \psi$.

A probability measure μ defined on the pseudo-borel subsets of \mathcal{H} is a hidden mixed state if $\sigma(\chi_L) = \sigma(\chi_M)$ implies $\mu(L) = \mu(M)$. If μ is a hidden mixed state and μ' is a measure taking the same values of μ on the hidden propositions then also μ' is also a hidden mixed state.

The measure defined by $\mu(A) = \eta_{\mathbb{C} \cdot \psi}(A \cap \mathbb{C} \cdot \psi)$ on the pseudoborel subsets of \mathcal{H} (where $\psi \neq 0$) is a hidden mixed state. If $\{\mu_k\}_{k \geq 1}$ is a sequence of hidden mixed states and $\{w_k\}_{k \geq 1}$ is a sequence of real numbers in $[0, 1]$ with $\sum_k w_k = 1$ then $\sum_k w_k \cdot \mu_k$ is a hidden mixed state.

Lemma 1. Let $D = \sum_k w_k \cdot E_{\mathbb{C} \cdot \psi_k}$ be a density matrix (where $\{\psi_k\}_{k \geq 1}$ is an orthonormal basis of \mathcal{H} and $\{w_k\}_{k \geq 1}$ is a sequence of real numbers in $[0, 1]$ with $\sum_k w_k = 1$) for every bounded self-adjoint operator T we have:

$$\text{Trace}[T \cdot D] = \sum_k w_k \cdot \langle T \rangle_{\psi_k}$$

Proof. It is enough to compute the trace using the orthonormal basis $\{\psi_k\}_{k \geq 1}$. \square

Theorem 9. *Let μ be a probability measure on \mathcal{H} , the measure μ is a hidden mixed state on \mathcal{H} if and only if there exists exactly one density matrix D on \mathcal{H} such that: $\mu(L) = \text{Trace}[\sigma(\chi_L) \cdot D]$ for every hidden proposition L .*

Proof. (\Leftarrow) If $D = \sum_k w_k \cdot E_{\mathbb{C} \cdot \psi_k}$ is any density matrix (where $\{\psi_k\}_{k \geq 1}$ is an orthonormal basis of \mathcal{H} and $\{w_k\}_{k \geq 1}$ is a sequence of real numbers in $[0, 1]$ with $\sum_k w_k = 1$) then the examples above show that $\mu'(A) = \sum_k w_k \cdot \eta_{\mathbb{C} \cdot \psi_k}(A \cap \mathbb{C} \cdot \psi_k)$ (where A is any pseudo-borel subset of \mathcal{H}) is a hidden mixed; since we have $\mu'(L) = \sum_k w_k \cdot \langle \sigma(\chi_L) \rangle_{\psi_k} = \text{Trace}[\sigma(\chi_L) \cdot D] = \mu(L)$ for every hidden proposition L the measure μ is also a hidden state.

(\Rightarrow) Since $\mu(L) = \mu(M)$ whenever $\sigma(\chi_L) = \sigma(\chi_M)$ it is well defined a map $\hat{\mu} : PR(\mathcal{H}) \rightarrow [0, 1]$ on the space $PR(\mathcal{H})$ of orthogonal projection operators by $\hat{\mu}(E) = \mu(L)$ for $\sigma(\chi_L) = E$. We will show that the map $\hat{\mu}$ is a measure on the closed subspaces of \mathcal{H} .

We need to prove that for every sequence $\{E_n\}$ of projectors with $E_n \cdot E_m = 0$ whenever $n \neq m$ we have $\hat{\mu}(\sum E_n) = \sum \hat{\mu}(E_n)$.

Let $a_n = 1 - 1/n$ for $n \geq 1$ and $a_0 = -\infty$, consider the spectral family defined by: $E(s) = 0$ if $s < 0$, $E(s) = E_1 + \dots + E_n$ if $a_n \leq s < a_{n+1}$ and $E(s) = I$ if $1 \leq s$.

The self-adjoint operator T defined by this family is bounded with $E_{[a_{n-1}, a_n]}^T = E_n$ for every $n \geq 1$; let f be an observable function with $\sigma(f) = T$.

Let $L_n = f^{-1}[a_{n-1}, a_n]$ for $n \geq 1$, every L_n is a hidden proposition with $\sigma(\chi_{L_n}) = E_n$. Let $L = f^{-1}[-\infty, 1[$, the set L is a hidden proposition disjoint union of all the $\{L_n\}$ with $\sigma(\chi_L) = \sum_{n \geq 1} E_n$. Then $\hat{\mu}(\sum_{n \geq 1} E_n) = \mu(L) = \sum_{n \geq 1} \mu(L_n) = \sum_{n \geq 1} \hat{\mu}(E_n)$.

Therefore for the Gleason's Theorem (cfr. [G]) there exists a (unique) density matrix D such that $\hat{\mu}(E) = \text{Trace}[E \cdot D]$ for every projector E . \square

We will denote by \mathcal{S} the family of all hidden mixed states on \mathcal{H} and by $DM(\mathcal{H})$ the set of all density matrices on \mathcal{H} , the previous theorem states there is a surjective map $\delta : \mathcal{S} \rightarrow DM(\mathcal{H})$ associating to a measure μ a density matrix $\delta(\mu)$ such that $\mu(L) = \text{Trace}[\sigma(\chi_L) \cdot \delta(\mu)]$ for every hidden proposition L .

Theorem 10. *For every hidden observable f and every hidden mixed state μ we have:*

$$\text{Trace}[\sigma(f) \cdot \delta(\mu)] = \int f \cdot d\mu$$

Proof. Let's write $T = \sigma(f)$ and $D = \delta(\mu)$. For every real number r the projector associated to the hidden proposition $f^{-1}(-\infty, r]$ is $\sigma(\chi_{f^{-1}(-\infty, r]}) = \chi_{(-\infty, r]} \circ T = E_{(-\infty, r]}^T$, therefore $\mu(f^{-1}(-\infty, r]) = \text{Trace}[E_{(-\infty, r]}^T \cdot D]$.

Remembering the properties of ch. 8 in [K and S] we have: $\int_{\mathcal{H}} f \cdot d\mu = \int_{\mathbb{R}} id \cdot df_* \mu = \int_{\mathbb{R}} id \cdot d\nu_F$ where $F(r) = f_* \mu(-\infty, r] = \mu(f^{-1}(-\infty, r]) = \text{Trace}[E_{(-\infty, r]}^T \cdot D]$.

When $D = E_{\mathbb{C} \cdot \psi_k}$ we get $F_k(r) = \text{Trace}[E_{(-\infty, r]}^T \cdot E_{\mathbb{C} \cdot \psi_k}] = \langle E_{(-\infty, r]}^T \rangle_{\psi_k}$ and $\int_{\mathcal{H}} f \cdot d\mu = \int_{\mathbb{R}} id \cdot d\nu_{F_k} = \langle T \rangle_{\psi_k} = \text{Trace}[T \cdot E_{\mathbb{C} \cdot \psi_k}]$.

For a general $D = \sum_k w_k \cdot E_{\mathbb{C} \cdot \psi_k}$ (where $\{\psi_k\}_{k \geq 1}$ is an orthonormal basis of \mathcal{H} and $\{w_k\}_{k \geq 1}$ is a sequence of real numbers in $[0, 1]$ with $\sum_k w_k = 1$) we have

$F(r) = \sum_k w_k \cdot \left\langle E_{(-\infty, r]}^T \right\rangle_{\psi_k}$, that is: $F = \sum_k w_k \cdot F_k$, and $\nu_F = \sum w_k \cdot \nu_{F_k}$, therefore: $\text{Trace}[T \cdot D] = \sum_k w_k \cdot \text{Trace}[T \cdot E_{\mathbb{C} \cdot \psi_k}] = \sum_k w_k \cdot \langle T \rangle_{\psi_k} = \sum_k w_k \cdot \int_{\mathbb{R}} id \cdot d\nu_{F_k} = \int_{\mathbb{R}} id \cdot d\nu_F$ and this proves the equality. \square

Corollary 2. *For every hidden observable f , every hidden mixed state μ and every b in \mathcal{B} we have :*

$$\text{Trace}[b \circ \sigma(f) \cdot \delta(\mu)] = \int b \circ f \cdot d\mu$$

Proof. Apply the previous theorem to $b \circ f$. \square

5. A UNIQUENESS THEOREM

Definition 5. *A theory with hidden variables (relative to a Hilbert space \mathcal{H}) is given assigning:*

- a set Λ (the hidden variables space)
- a surjective map $\pi : \Lambda \rightarrow \mathcal{P}(\mathcal{H})$ (defining for each $[\psi]$ in $\mathcal{P}(\mathcal{H})$ the fiber $\Lambda_{[\psi]} = \pi^{-1}([\psi])$)
- on each fiber $\Lambda_{[\psi]}$ a σ -algebra $\mathcal{M}_{[\psi]}$ of subsets and a measure $\mu_{[\psi]}$ making $\Lambda_{[\psi]}$ a standard non-atomic probability space
- a set \mathcal{G} of real functions on Λ (the hidden observables) pseudo-measurables (that is $f^{-1}(B) \cap \Lambda_{[\psi]}$ is a measurable subset of $\Lambda_{[\psi]}$ for every f in \mathcal{G} and every borel subset B in the real line) and essentially bounded (that is each f is bounded out of a suitable subset N_f of Λ with $\mu_{[\psi]}(N_f \cap \Lambda_{[\psi]}) = 0$ for each $[\psi]$ in $\mathcal{P}(\mathcal{H})$)
- a surjective map $\beta : \mathcal{G} \rightarrow \mathcal{B}_{sa}(\mathcal{H})$

such that: $\mu_{[\psi]}(f^{-1}(B) \cap \Lambda_{[\psi]}) = \left\langle E_B^{\beta(f)} \right\rangle_{\psi}$ for every f in \mathcal{G} , every borel subset B in the real line and every $[\psi]$ in $\mathcal{P}(\mathcal{H})$.

- Obviously the datum of $\mathcal{H} = \mathcal{H} \setminus \{0\}$, of the canonical map $q : \mathcal{H} \rightarrow \mathcal{P}(\mathcal{H})$, of the sets $\mathbb{C}_{[\psi]} = (\mathbb{C} \setminus \{0\}) \cdot \psi$ with the measures $\eta_{[\psi]}$ and the set of functions \mathcal{O} with the map σ defined in the previous sections is a hidden variable theory
- For simplicity we consider on the fibers $\Lambda_{[\psi]}$ the most natural structure of probability space; moreover the functions in \mathcal{G} are taken essentially bounded otherwise we would need to deal with non-bounded self-adjoint operators.
- two pseudo-measurables functions on Λ will be considered equal if they coincide out of a pseudo-measurable subset of Λ .

Definition 6. *Two pseudo-measurable and essentially bounded functions f_1 and f_2 on Λ will be called statistically equivalent if $\mu_{[\psi]}(f_1^{-1}(B) \cap \Lambda_{[\psi]}) = \mu_{[\psi]}(f_2^{-1}(B) \cap \Lambda_{[\psi]})$ for every borel subset B in the real line and every $[\psi]$ in $\mathcal{P}(\mathcal{H})$; the family \mathcal{G} of a hidden variable theory will be called maximal if whenever \mathcal{G} contains a function it contains also all its statistically equivalent functions.*

- if f_1 and f_2 in \mathcal{G} are statistically equivalent then $\beta(f_1) = \beta(f_2)$

- the family \mathcal{G} can always be extended to a maximal family $\tilde{\mathcal{G}}$ and the map β can be extended in a unique way to a map $\tilde{\beta}$ in such a way to have the same value on statistically equivalent functions. Considering this new family $\tilde{\mathcal{G}}$ instead of \mathcal{G} and the map $\tilde{\beta}$ instead of β we get a new hidden variables theory.

Theorem 11. *The family \mathcal{O} is maximal.*

Proof. Let f be a function in \mathcal{O} and let g be a (pseudo-measurable and essentially bounded) function on \mathcal{H} statistically equivalent to f ; to prove that g is also in \mathcal{O} we have to show that g has horthodox mean values and that each $g^{-1}(B)$ is a hidden proposition for every borel subset B in the real line.

Let $T = \sigma(f)$, fixed ψ in \mathcal{H} let's define $F : \mathbb{R} \rightarrow [0, 1]$ by $F(r) = \langle E_{(-\infty, r]}^T \rangle_\psi$. Since T is bounded the measure ν_F has support inside a suitable bounded interval. Denoted by $f|_{\mathbb{C}_{[\psi]}}$ and $g|_{\mathbb{C}_{[\psi]}}$ the restrictions of f and g to $\mathbb{C}_{[\psi]}$ we have $g|_{\mathbb{C}_{[\psi]}} \eta_{[\psi]} = f|_{\mathbb{C}_{[\psi]}} \eta_{[\psi]} = \nu_F$; because the identity function in \mathbb{R} is absolutely integrable with respect to ν_F the function $g|_{\mathbb{C}_{[\psi]}}$ is absolutely integrable with respect to $\eta_{[\psi]}$ (cfr. Cor. 3 pag. 93 of [K and S]) and $\int_{\mathbb{C}_{[\psi]}} g|_{\mathbb{C}_{[\psi]}} \cdot d\eta_{[\psi]} = \int_{\mathbb{R}} id \cdot d\nu_F = \int_{\mathbb{R}} id \cdot d\nu_{\langle E_{(-\infty, r]}^T \rangle_\psi} = \langle T \rangle_\psi$. This proves that g has horthodox mean values.

Fixed a borel subset B in the real line since the characteristic function $\chi_{g^{-1}(B)}$ has mean values: $\int_{\mathbb{C}_{[\psi]}} \chi_{g^{-1}(B)} \cdot d\eta_{[\psi]} = \eta_{[\psi]}(f^{-1}(B) \cap \mathbb{C}_{[\psi]}) = \langle E_B^T \rangle_\psi$ given by the projector E_B^T the set $g^{-1}(B)$ is a hidden proposition. \square

Definition 7. *An isomorphism (mod 0) between two hidden variables theories $(\Lambda, \pi, \mu, \mathcal{G}, \beta)$ and $(\Lambda', \pi', \mu', \mathcal{G}', \beta')$ is given by a map $\Phi : \Lambda \setminus N \rightarrow \Lambda' \setminus N'$ (where N and N' are pseudo-measurable null subsets respectively of Λ and Λ') with the following properties:*

- Φ is bijective
- $\pi = \pi' \circ \Phi$ (and therefore $\Phi(\Lambda_{[\psi]}) \subset \Lambda'_{[\psi]}$ for every $[\psi]$ in $\mathcal{P}(\mathcal{H})$)
- $\Phi|_{\Lambda_{[\psi]} \setminus N} : \Lambda_{[\psi]} \setminus N \rightarrow \Lambda'_{[\psi]} \setminus N'$ is a measure preserving borel equivalence ($\Phi|_{\mathbb{C}_{[\psi]}} \mu_{[\psi]} = \mu'_{[\psi]}$) for every $[\psi]$ in $\mathcal{P}(\mathcal{H})$
- $\Phi_* \mathcal{G} \subset \mathcal{G}'$ (where $\Phi_*(f) = f \circ \Phi^{-1}$) and $\Phi_* : \mathcal{G} \rightarrow \mathcal{G}'$ is a bijective map

Note that an isomorphism (mod 0) $\Phi : \Lambda \setminus N \rightarrow \Lambda' \setminus N'$ automatically verifies the condition: $\beta = \beta' \circ \Phi_*$. In fact taken f in \mathcal{G} let $f' = \Phi_*(f)$, $T = \beta(f)$ and $T' = \beta'(f')$ we have: $\langle E_B^T \rangle_\psi = \mu_{[\psi]}(f^{-1}(B) \cap \Lambda_{[\psi]}) = \mu'_{[\psi]}((f')^{-1}(B) \cap \Lambda'_{[\psi]}) = \langle E_B^{T'} \rangle_\psi$ for every $[\psi]$ in $\mathcal{P}(\mathcal{H})$ and every borel subset B in the real line, therefore $T = T'$.

Theorem 12. *Two hidden variables theories with maximal spaces of hidden observables are isomorphic (mod 0).*

Proof. For each $[\psi]$ in $\mathcal{P}(\mathcal{H})$ since $\Lambda_{[\psi]}$ and $\Lambda'_{[\psi]}$ are standard non-atomic probability spaces there exists an isomorphism (mod 0) (a measure preserving borel equivalence) $\Phi_{[\psi]} : \Lambda_{[\psi]} \setminus N_{[\psi]} \rightarrow \Lambda'_{[\psi]} \setminus N'_{[\psi]}$ (where $N_{[\psi]}$ and $N'_{[\psi]}$ are measurable null subsets respectively of $\Lambda_{[\psi]}$ and $\Lambda'_{[\psi]}$). Taken $N = \bigcup N_{[\psi]}$ and $N' = \bigcup N'_{[\psi]}$ and

defined $\Phi : \Lambda \setminus N \rightarrow \Lambda' \setminus N'$ by $\Phi(\lambda) = \Phi_{[\psi]}(\lambda)$ when λ is in $\Lambda_{[\psi]} \setminus N_{[\psi]}$ we get a map verifying the first three conditions of an isomorphism (mod 0).

Taken f in \mathcal{G} the function $f' = \Phi_*(f)$ is pseudo-measurable and essentially bounded on Λ' , considered $T = \beta(f)$ and choosen g' in \mathcal{G}' such that $\beta'(g') = T$ let's prove that f' is statistically equivalent to g' , the maximality of \mathcal{G}' will imply then that also f' is in \mathcal{G}' .

We have in fact: $\mu'_{[\psi]}((g')^{-1}(B) \cap \Lambda'_{[\psi]}) = \langle E_B^{T'} \rangle_{\psi} = \mu_{[\psi]}(f^{-1}(B) \cap \Lambda_{[\psi]}) = \mu'_{[\psi]}((f')^{-1}(B) \cap \Lambda'_{[\psi]})$ for every $[\psi]$ in $\mathcal{P}(\mathcal{H})$ and every borel subset B in the real line. For the generality of f this means: $\Phi_*\mathcal{G} \subset \mathcal{G}'$, in an analogous way we can prove that $(\Phi^{-1})_*\mathcal{G}' \subset \mathcal{G}$. Since $\Phi_* \circ (\Phi^{-1})_* = id_{\mathcal{G}'}$ and $(\Phi^{-1})_* \circ \Phi_* = id_{\mathcal{G}}$ the map Φ_* is bijective. \square

Therefore every hidden variables theory is isomorphic (mod 0) to the theory $(\mathcal{H}, q, \eta., \mathcal{O}, \sigma)$ developped in the previous sections if its space of hidden observables is maximal otherwise it is isomorphic (mod 0) to a theory $(\mathcal{H}, q, \eta., \mathcal{O}', \sigma)$ with $\mathcal{O}' \subset \mathcal{O}$.

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